Avijit
Lectures on Dirac delta (introduction)

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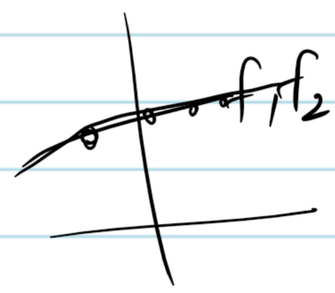
$\delta(x)$ distribution

$D_1(x)$ $D_2(x)$

$\phi(x)$

$\langle D, \phi \rangle$

$f_1 = f_2$



$D_1(x) = D_2(x)$ means $\langle D_1(x), \phi(x) \rangle = \langle D_2(x), \phi(x) \rangle$ for any test function ϕ.

Equality of distribution / Generalized functions :

→ Two distributions $D_1(x)$ & $D_2(x)$ are considered equal iff $\langle D_1(x), \phi(x) \rangle = \langle D_2(x), \phi(x) \rangle$ for every test function $\phi(x)$.

→ $\int_{-\infty}^{\infty} D_1(x)\,\phi(x)\,dx = \int_{-\infty}^{\infty} D_2(x)\,\phi(x)\,dx$

$\langle D, \phi \rangle$

$\langle \delta, \phi \rangle = \phi(0)$

$\simeq$

$\int_{-\infty}^{\infty} D(x)\phi(x)dx$

mental shortcut

$\langle D', \phi \rangle$

$\langle D, \phi \rangle$

In mathematical analysis, the **Dirac delta function** (or δ **distribution**), also known as the **unit impulse**,[1] is a generalized function on the real numbers, whose value is zero everywhere except at zero, and whose integral over the entire real line is equal to one.[2][3][4] Thus it can be represented heuristically as

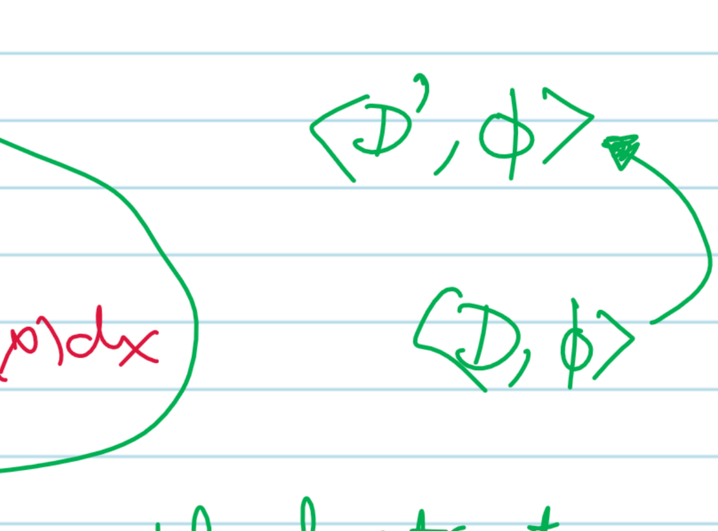
$$\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases}$$

such that

$$\int_{-\infty}^{\infty} \delta(x)dx = 1.$$

$\langle \delta(x), \phi(x) \rangle = \phi(0)$

axiom



Schematic representation of the Dirac delta function by a line surmounted by an arrow. The height of the arrow is usually meant to specify the value of any multiplicative constant, which will give the area under the function. The other convention is to write the area next to the arrowhead.

Since there is no function having this property, modelling the delta "function" rigorously involves the use of limits or, as is common in mathematics, measure theory and the theory of distributions.

Question

Why do we use integrals to define actions of a disribution on a test function?

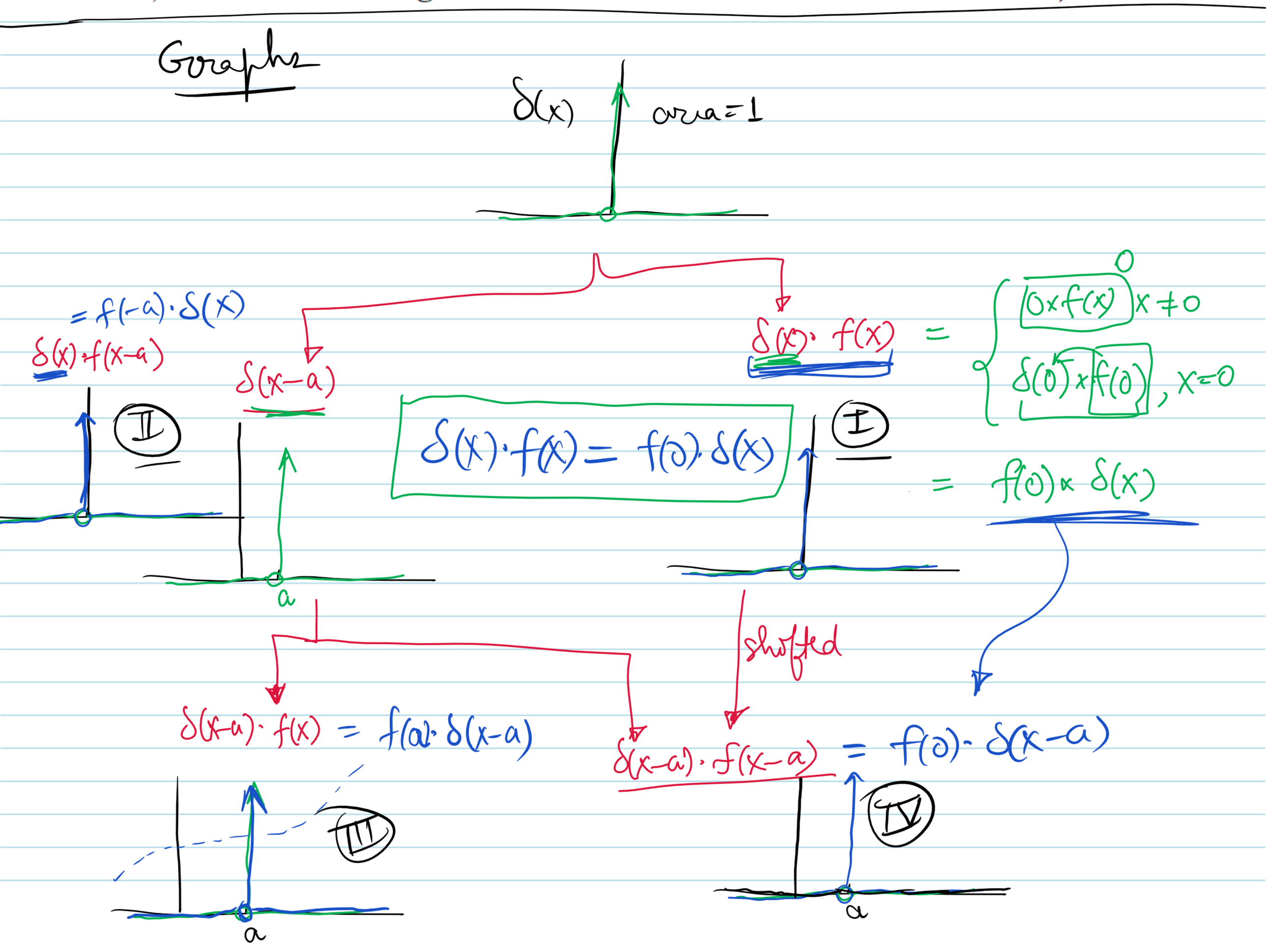
Answer

Basically, this integral notation is a mental shortcut that extends the rules of calculus. It allows us to define operations (like derivatives) on objects which are not functions in the classical sense, but because of this notation, they behave as if they follow the rules of calculus.

We, the mathematicians, designed the operations on distributions (like the derivative $\langle T', \Phi \rangle = -\langle T, \Phi' \rangle$) specifically so that when you look at the notation and the manipulation, it appears as if the same old calculus is happening. We created the illusion to be perfect.

(Basically, yeh integral notation ek mental shortcut hai jo calculus ke rules ko extend karta hai. Ye hume un objects par bhi operations (like derivative) define karne deta hai jo asal mein functions nahi hain, lekin is notation ki wajah se dekhne me lagta hai ki wo calculus ke rules follow karte hain.)

Graphs



$$\left[\langle \delta(x), \phi(x) \rangle = \phi(0) \right]$$ NO proof X

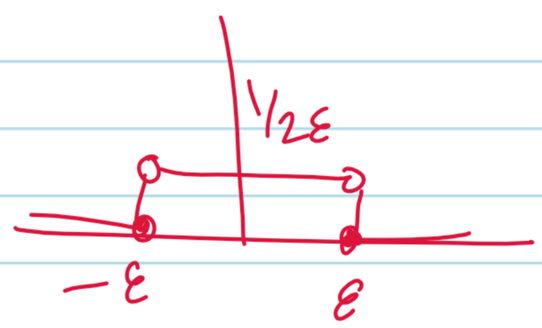
using the integral shortcut (heuristic representation)

$$\approx \int_{-\infty}^{\infty} \underbrace{\delta(x)\cdot\phi(x)}_{=\phi(0)\cdot\delta(x)} dx = \int_{-\infty}^{\infty} \phi(0)\,\delta(x)\,dx$$

$$= \phi(0)\left[\int_{-\infty}^{\infty} \delta(x)\,dx\right] \to 1$$

$$= \phi(0)$$

$$\delta_\varepsilon(x) = \begin{cases} \frac{1}{2\varepsilon}, & |x| < \varepsilon \\ 0, & \text{else} \end{cases}$$

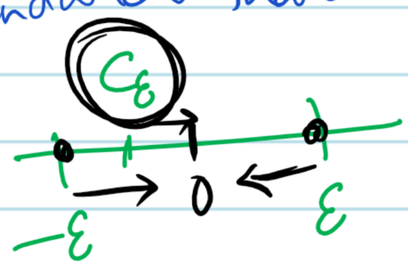


$$\lim_{\varepsilon \to 0} \delta_\varepsilon(x) = \delta(x)$$

$$\int_{-\infty}^{\infty} \delta(x)\,\phi(x)\,dx = \phi(0).$$

$$I_\varepsilon = \int_{-\infty}^{\infty} \delta_\varepsilon(x)\,\phi(x)\,dx = \int_{-\varepsilon}^{\varepsilon} \frac{\phi(x)}{2\varepsilon}\,dx$$

[sandwich theorem]



$$= \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} \phi(x)\,dx \quad \text{def. } \int \text{ MVT}$$

$$= \frac{1}{2\varepsilon} \times \left(\phi(c_\varepsilon) \times 2\varepsilon\right)$$

for some c_ε:

$$\Rightarrow I_\varepsilon = \phi(c_\varepsilon)$$

$$-\varepsilon \le c_\varepsilon \le \varepsilon$$

where $-\varepsilon \le c_\varepsilon \le \varepsilon$.

$$\lim_{\varepsilon \to 0} I_\varepsilon = \lim_{\varepsilon \to 0} \phi(c_\varepsilon) = \phi\left(\lim_{\varepsilon \to 0}(c_\varepsilon)\right) = \phi(0)$$

($\lim_{\varepsilon \to 0}(c_\varepsilon) \to 0$)

$$\Rightarrow \boxed{\lim_{\varepsilon\to 0}} \int_{-\infty}^{\infty} \delta_\varepsilon(x)\phi(x)dx = \underline{\phi(0)}$$

this info was not needed of lim is outside.

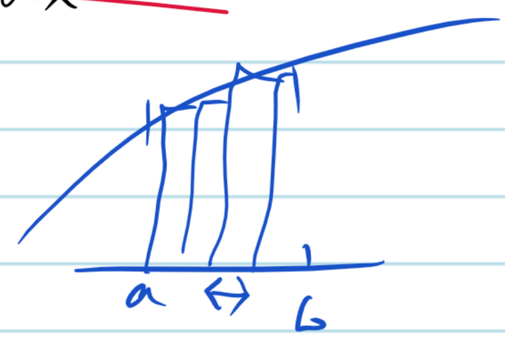
$$\int_{-\infty}^{\infty} \delta(x)dx = 1$$

$$\Rightarrow \underbrace{\langle \delta(x), \phi(x)\rangle = \phi(0)} \approx \int_{-\infty}^{\infty} \delta(x)\phi(x)dx$$

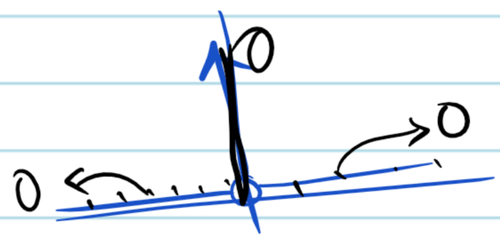
(True)

$$\neq \int_{-\infty}^{\infty} \left(\lim_{\varepsilon\to 0} \delta_\varepsilon(x)\right)\phi(x)dx$$

$$\boxed{\delta(x)} = \phi(0)\int_{-\infty}^{\infty}\delta(x)dx \to 1$$



$$= 0 \quad (\text{old calculus})$$



$= \phi(0) \times 1$, where $\int_{-\infty}^{\infty}\delta(x)dx$ is defined as 1.

$\langle \delta(x), \phi(x)\rangle = \phi(0)$ is a fact.

So, $\langle \delta(x), \phi(x)\rangle$ can be thought of as $\boxed{\lim_{\varepsilon\to 0}} \int_{-\infty}^{\infty} \delta_\varepsilon(x)\phi(x)dx$. "$\langle \delta(x), \phi(x)\rangle$"

* But lim can NOT be taken inside.

Another inconsistency arises when $\int_{-\infty}^{\infty}\delta(x)dx$ is defined as 1.

$$\int_{-\infty}^{\infty}\delta(x)dx \longrightarrow \langle \delta(x), 1\rangle$$

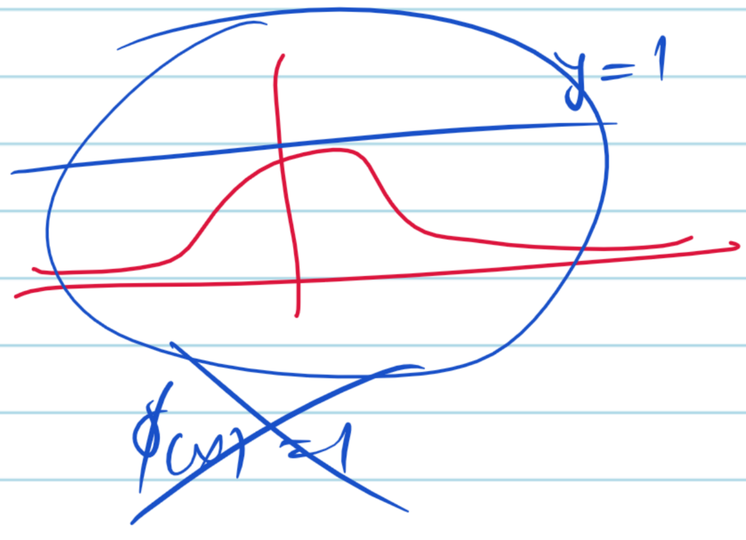
$= 1$

but 1 IS NOT a test function.

$$\int_{-\infty}^{\infty} 1\cdot\delta(x)dx = 1$$

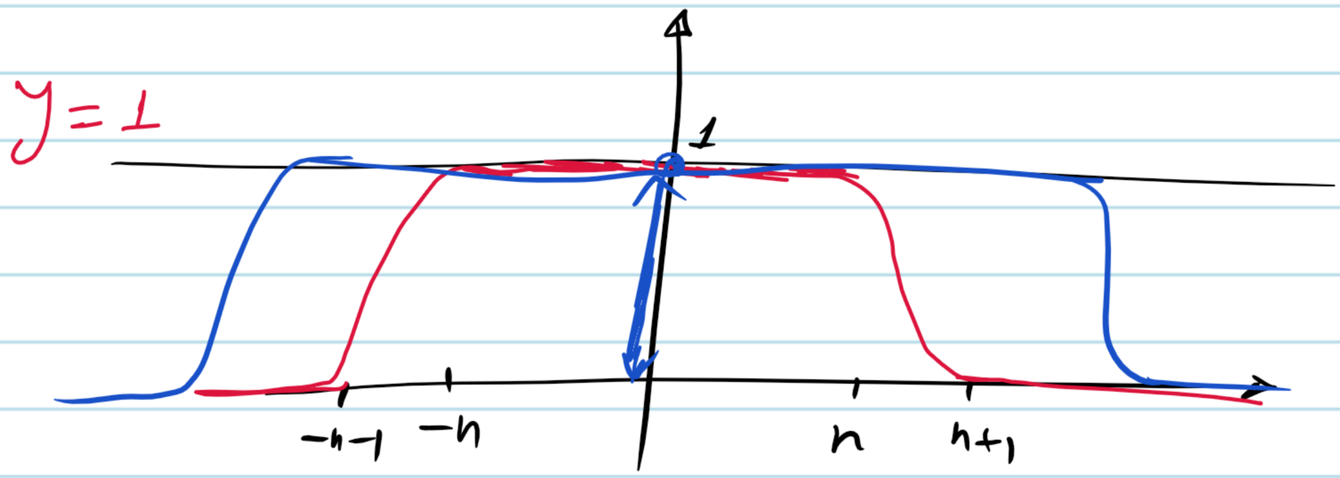
$\phi(x)'' = \phi(0)$

$\langle \delta(x), 1\rangle = 1$



How to deal with $\langle \delta(x), \boxed{1} \rangle = 1$?

$$\int_{-\infty}^{\infty} \delta(x)\,dx = 1$$



$1 \in \mathcal{D}$

$\phi_n(x) \in \mathcal{D}$

$n \to \infty \quad \phi_n(0) = 1$

$$\langle \delta(x), \phi_n(x) \rangle = \phi_n(0) = 1$$

$$\lim_{n\to\infty} \langle \delta(x), \phi_n(x) \rangle = 1$$

$$\langle \delta(x), 1 \rangle := \lim_{n\to\infty} \langle \delta(x), \boxed{\phi_n(x)} \rangle$$

{ := or def }

Product of distribution T with function f :

$$\langle f \cdot T, \phi \rangle = \langle T, f \cdot \phi \rangle$$

$T \times f$

e.g. $x \cdot \delta(x) = ?$

$f \times T = T$ ✗

$$\langle x\,\delta(x), \phi \rangle = \langle \delta(x), x \cdot \phi(x) \rangle$$

$$= 0 \times \phi(0) \overset{\text{scalar}}{=} 0 = \langle 0_F, \phi \rangle$$

$$\langle x\,\delta(x), \phi \rangle = \langle 0_F, \phi \rangle$$

$$\Rightarrow \boxed{x \cdot \delta(x) = 0_F}$$

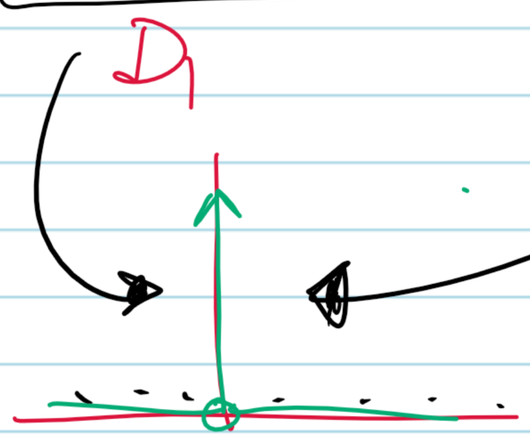
on shifting,
$(x-a) \cdot \delta(x-a) = 0_F$

Equality of distributions T_1 & T_2 (Recap)

$$T_1 = T_2 \iff \langle T_1, \phi \rangle = \langle T_2, \phi \rangle \quad \forall\ \phi \in \mathcal{D}$$

Shortcut: $\langle T, \phi \rangle = \int_{-\infty}^{\infty} T(x)\phi(x)\,dx$

(Q.1) $\underline{2\cdot\delta(x)}$ and $\underline{3\cdot\delta(x)}$. (equality ?)



different area

	area
$2\delta(x)$	2
$3\delta(x)$	3

$\langle 2\delta(x), \phi \rangle = 2 \cdot \langle \delta(x), \phi \rangle = 2\phi(0)$

$\int 2\delta(x)\phi(x)$

$\langle 3\delta(x), \phi \rangle = 3 \cdot \langle \delta(x), \phi \rangle = 3\phi(0)$

$2\delta(x) \neq 3\delta(x)$

$D_1 \neq D_2$

1) same graph $\not\Rightarrow$ same distribution

2) same distribution $\Rightarrow$ same action on every test function.

(Q.2) $\delta(x^2 - 4) =$?

δ(linear)

(Q2) Show that $D_1 = D_2$ using the heuristic representation.

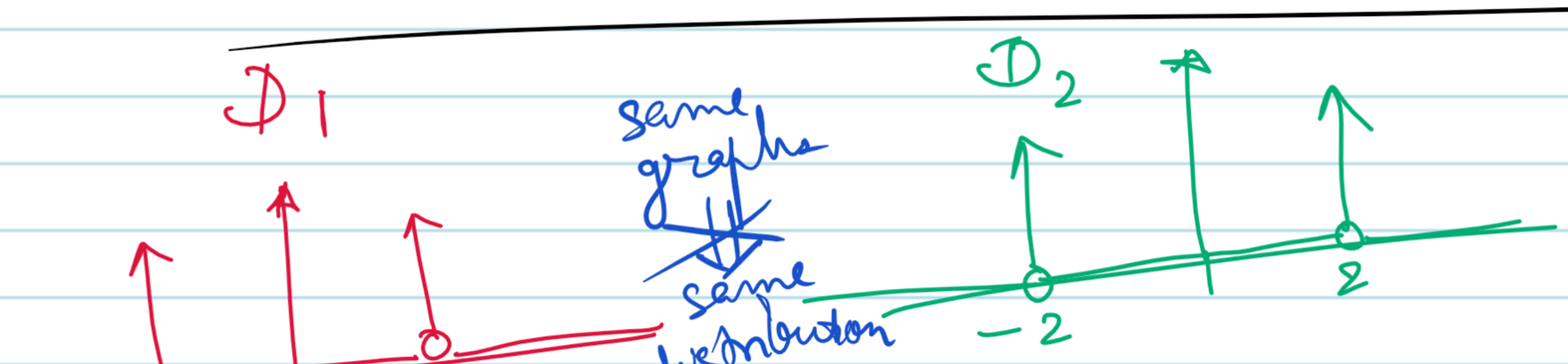
(i.e. → ∫ – shortcut.)

$\Rightarrow D_1(x) = \delta(x^2-4)$

$\Rightarrow D_2(x) = \frac{1}{4}\delta(x-2) + \frac{1}{4}\delta(x+2)$

OR

$\langle D_1(x), \phi \rangle = \langle D_2(x), \phi \rangle \quad \forall \phi$.



from graph, $\delta(x^2-4) = c_1\delta(x-2) + c_2\delta(x+2)$

where $c_1>0$ & $c_2>0$.

$\langle D_1, \phi(x) \rangle$

$= \langle \delta(x^2-4), \phi(x) \rangle = \int_{-\infty}^{\infty} \delta(x^2-4)\,\phi(x)\,dx$

$= \int_{-\infty}^{0} \delta(x^2-4)\phi(x)\,dx + \int_{0}^{\infty} \delta(x^2-4)\phi(x)\,dx$

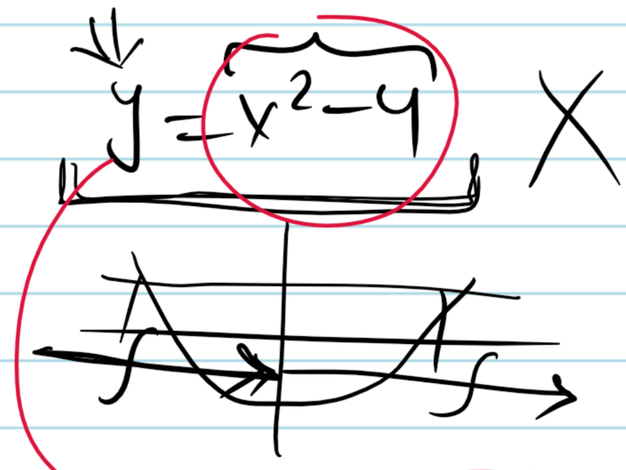
$x<0 \quad x = -\sqrt{y+4}$

$x>0 \quad x = \sqrt{y+4}$

$y = x^2-4$

$\frac{dy}{dx} = 2(-\sqrt{y+4})$

$= \int_{\infty}^{-4} \delta(y)\,\phi(-\sqrt{y+4})\frac{dy}{-2\sqrt{y+4}} + \int_{-4}^{\infty} \delta(y)\,\phi(\sqrt{4+y})\frac{dy}{2\sqrt{y+4}}$



$x = \pm\sqrt{y+4}$

$= \frac{1}{2}\int_{-4}^{\infty} \delta(y)\frac{\phi(-\sqrt{y+4})}{\sqrt{y+4}}\,dy + \frac{1}{2}\int_{-4}^{\infty} \delta(y)\frac{\phi(\sqrt{y+4})}{\sqrt{y+4}}\,dy$

any -ve real number → -4 → $-\infty$

$= \frac{1}{2}\left\langle \delta(y), \frac{\phi(-\sqrt{y+4})}{\sqrt{y+4}} \right\rangle + \frac{1}{2}\left\langle \delta(y), \frac{\phi(\sqrt{y+4})}{\sqrt{y+4}} \right\rangle$

$= \frac{1}{2}\times\frac{\phi(-2)}{2} + \frac{1}{2}\times\frac{\phi(2)}{2} = \frac{1}{4}\phi(-2) + \frac{1}{4}\phi(2)$

$\langle \delta(x), \phi(x-2) \rangle \quad \langle \delta(x+2), \phi(x) \rangle$

So, $\langle \delta(x^2-4), \phi(x) \rangle = \frac{1}{4} \langle \delta(x+2), \phi(x) \rangle + \frac{1}{4} \langle \delta(x-2), \phi(x) \rangle$

$\langle T_1, \phi \rangle + \langle T_2, \phi \rangle = \langle T_1 + T_2, \phi \rangle$

Addition of distributions
[ϕ should be same
using continuous linear functional
(OR)
Integration shortcut]

$= \left\langle \frac{1}{4}\delta(x+2), \phi(x) \right\rangle + \left\langle \frac{1}{4}\delta(x-2), \phi(x) \right\rangle$

$= \left\langle \frac{1}{4}\delta(x+2) + \frac{1}{4}\delta(x-2), \phi(x) \right\rangle, \quad \forall \phi(x) \in \mathcal{D}$

$\therefore$

$\delta(x^2-4) = \frac{1}{4}\delta(x-2) + \frac{1}{4}\delta(x+2)$

Q.E.D.

Distributions [RECAP]

$\begin{cases} T_1, T_2 = \text{distributions} \in \mathcal{D}' \\ \phi = \text{test function} \in \mathcal{D} \end{cases}$

(R-1) **Equality:** $\langle T_1, \phi \rangle = \langle T_2, \phi \rangle \quad \forall \phi \iff T_1 = T_2$

(R-2) **Shortcut Notation / Distributional notation / Heuristic Representation**
(used by Engg./Physicists as a mental shortcut)

* action $\neq$ integration but calculations match

* $\langle T(x), \phi(x) \rangle = \int_{-\infty}^{\infty} T(x) \cdot \phi(x)\, dx$

(R-3) **Distributions are linear continuous functionals**

(i) $\langle T_1, \phi \rangle + \langle T_2, \phi \rangle = \langle T_1 + T_2, \phi \rangle$

(ii) $\langle \lambda T, \phi \rangle = \lambda \cdot \langle T, \phi \rangle$

A linear functional $T : \mathcal{D}(\mathbb{R}^n) \to \mathbb{R}$ is called **continuous** if:

$$\{\phi_k\} \xrightarrow{\mathcal{D}} 0 \Rightarrow \{T(\phi_k)\} \to 0$$

That is, if a sequence of test functions converges to zero, then the sequence of their functionals also converges to zero.

(R-4) $\langle f \cdot T, \phi \rangle = \langle T, f \cdot \phi \rangle$ if f is a well behaved function.

** **(R-5)** **Action of derivative of T**

$$\Rightarrow \boxed{\langle T', \phi \rangle = -\langle T, \phi' \rangle}$$

$\langle T, \phi \rangle$ = ✓

$\langle T', \phi \rangle$ = ?

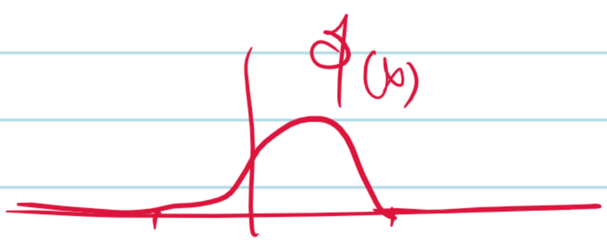
Why?

$$\langle T', \phi \rangle = \int_{-\infty}^{\infty} \underset{II}{T'(x)}\, \underset{I}{\phi(x)}\, dx = \Big(\phi(x) \cdot T(x)\Big)_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \phi'(x) \cdot T(x)\, dx$$

$=$

$\phi(\infty) \times$ — $= 0$ $\qquad$ $\phi(-\infty) \times$ — $= 0$

$= 0 - \langle T(x), \phi'(x) \rangle$



* If T is distribution, then T' is also distribution. (can be proved using (R-3))

Given $\begin{cases} T_1, T_2, T \longrightarrow \text{distributions} \\ \phi_1, \phi_2, \phi \longrightarrow \text{test functions} \\ \alpha, \beta, \cdots \longrightarrow \text{Scalars} \end{cases}$

Show that : ① $(T_1+T_2)' = T_1' + T_2'$

② $(\alpha T)' = \alpha \cdot T'$

③ n-derivatives $\langle T^{(n)}, \phi \rangle = (-1)^n \cdot \langle T, \phi \rangle$

④ product with ∞ly diff. function ψ $(T \cdot \psi)' = T'\psi + T\psi'$

ψ need not to be a test function. It should just be ∞ly differentiable.

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① $\langle (T_1+T_2)', \phi \rangle = -\langle T_1+T_2, \phi' \rangle$

$= -\langle T_1, \phi' \rangle - \langle T_2, \phi' \rangle$

$= \langle T_1', \phi \rangle + \langle T_2', \phi \rangle = \langle T_1' + T_2', \phi \rangle$

$(T_1+T_2)' = T_1' + T_2'$

② $\langle (\alpha \cdot T)', \phi \rangle = -\langle \alpha T, \phi' \rangle = -\alpha \langle T, \phi' \rangle$

$= \alpha \cdot \langle T', \phi \rangle$

$= \langle \alpha T', \phi \rangle$

$(\alpha T)' = \alpha \cdot T'$

③ $\langle T^{(n)}, \phi \rangle = (-1)^n \cdot \langle T, \phi^{(n)} \rangle$ ✓

$\langle T', \phi \rangle = -\langle T, \phi'' \rangle$

$\langle T'', \phi \rangle = +\langle T, \phi'' \rangle$

$T^{(n)}$ or $T^{\langle n \rangle} = \dfrac{d^n}{dx^n}(T(x))$

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④ $\langle (\psi T)', \phi \rangle = -\langle \psi T, \phi' \rangle$ using $\langle fT, \phi \rangle = \langle T, f\phi \rangle$

$= -\langle T, \psi \cdot \phi' \rangle$

product rule: $(\phi \cdot \psi)' = \phi' \psi + \phi \psi' \Rightarrow \psi \cdot \phi' = (\phi\psi)' - \phi\psi'$

$= -\langle T, (\phi\psi)' - \phi\psi' \rangle = -\langle T, (\phi\psi)' \rangle + \langle T, \phi\psi' \rangle$

(use shortcut notations)

$$\langle (T\psi)', \phi \rangle = \langle T, \phi\psi' \rangle - \langle T, (\phi\psi)' \rangle$$

$$= \langle T, \phi(\psi') \rangle + \langle T, \phi(\psi) \rangle$$

again $\quad \langle (f)T, \phi \rangle = \langle T, (f)\phi \rangle$

$$= \langle \psi' T, \phi \rangle + \langle \psi T', \phi \rangle$$

$$= \langle \psi' T + \psi T', \phi \rangle \qquad \forall \phi$$

$$\therefore, \quad (\psi \cdot T)' = \psi' T + \psi T' \qquad Q.E.D.$$

Basic properties of Dirac delta function

$\delta \in \mathcal{D}'$ Dirac delta function $\quad \phi \in \mathcal{D}$ test function $\quad$ such that $\quad \langle \delta(x), \phi(x) \rangle = \phi(0)$

Action = point evaluation

Shortcut

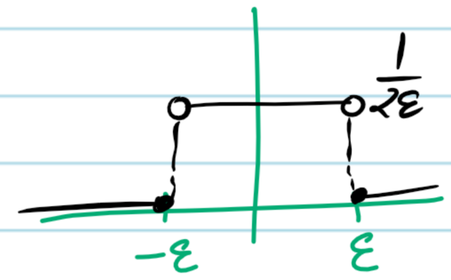
$$\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x)\, dx \overset{def}{=} 1$$

Alternative ways to write $\delta(x)$:

$\delta_\varepsilon(x)$

① $\delta(x) = \lim_{\text{width} \to 0} (\text{pulse}) = \lim_{\varepsilon \to 0} \delta_\varepsilon(x)$



② $\delta(x) = \int_{-\infty}^{\infty} e^{i(2\pi x)p}\, dp$

③ $\delta(x) = \frac{d}{dx}(H(x))$, $H(x)$ = Heaviside Step function.